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On the Approximation Properties of Cesàro Means of Negative Order of Walsh–Fourier Series

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In this paper we establish approximation properties of Cesàro $(C, -\alpha)$-means with $\alpha \in (0, 1)$ of Walsh–Fourier series. This result allows one to obtain the condition which is sufficient for the convergence of the means $\sigma_n^{-\alpha}(f, x)$ to $f(x)$ in the L^p-metric. We also show that this condition cannot be improved in the case $p = 1$.

Key Words: Walsh functions; Cesàro means; approximation.

1. INTRODUCTION

Let $r_0(x)$ be a function defined on $[0, 1)$ by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2) \\ -1, & \text{if } x \in [1/2, 1) \end{cases}, \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \qquad n \geqslant 1, \qquad \text{and} \qquad x \in [0, 1).$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e., $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s$ then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

The idea of using products of Rademacher's functions to define the Walsh system originated from Paley [6].

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The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

(1) $$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } \; x \in [0, 1/2^n), \\ 0, & \text{if } \; x \in [1/2^n, 1). \end{cases}$$

Suppose that f is a Lebesgue integrable function on $[0, 1]$ and 1-periodic. Then its Walsh–Fourier series is defined by

$$\sum_{k=0}^{\infty} \hat{f}(k) \, w_k(x),$$

where

$$\hat{f}(k) = \int_0^1 f(t) \, w_k(t) \, dt$$

is called the kth Walsh–Fourier coefficient of function f.

Denote the nth partial sum of the Walsh–Fourier series of the function f by $S_n(f, x)$:

$$S_n(f, x) = \sum_{k=0}^{n-1} \hat{f}(k) \, w_k(x).$$

The Cesàro (C, α)-means of the Walsh–Fourier series are defined as

$$\sigma_n^\alpha(f, x) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^\alpha \, \hat{f}(k) \, w_k(x),$$

where

$$A_0^\alpha = 1, \qquad A_n^\alpha = \frac{(\alpha+1) \cdots (\alpha+n)}{n!}, \quad \alpha \neq -1, -2, \ldots$$

It is well known that [11, Chap. 3]

(2) $$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1},$$

(3) $$A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1},$$

(4) $$A_n^\alpha \sim n^\alpha.$$

We remind the reader that $C_w([0,1])$ is the collection of functions $f\colon [0,1) \to R$ that are uniformly continuous from the dyadic topology of $[0,1)$ to the usual topology of R, or shortly W-continuous (see [7, pp. 9–11]).

Let $L^p([0,1])$ denote the collection of all measurable 1-periodic functions defined on $[0,1]$, with the following norms

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p} < \infty \qquad (1 \leqslant p < \infty).$$

In case $p = \infty$, by $L^p([0,1])$ we mean $C_w([0,1])$, endowed with the supremum norm.

Let $f \in L^p([0,1])$. The expression

$$\omega(\delta, f)_p = \sup_{0 \leqslant h < \delta} \|f(\cdot \oplus h) - f(\cdot)\|_p$$

is called the dyadic modulus of continuity, where $\oplus$ denotes dyadic addition [3, Chap. 1].

The problems of summability of Cesàro means of positive order for Walsh–Fourier series were studied in [1, 8].

Tevzadze [9] has studied the uniform convergence of Cesàro means of negative order of Walsh–Fourier series. In particular, in terms of moduli of continuity and variation of function $f \in C_w([0,1])$ he has proved the criterion for the uniform summability by the Cesàro method of negative order of Fourier series with respect to the Walsh system.

In [4, 5] the author proved conditions sufficient for the convergence of Cesàro means of negative order of Walsh–Fourier series in spaces $L^p([0,1])$, $1 \leqslant p \leqslant \infty$.

In [4, 5, 9] the results were established without estimation of approximation.

In his monography [10, Part 1, Chap. 4] Zhizhiashvili investigated the behavior of Cesàro means of negative order for trigonometric Fourier series in detail. In this paper we study analogical questions in case of the Walsh system.

Theorem 1. *Let $f(x)$ belong to $L^p([0,1])$ for some $p \in [1, \infty]$ and $\alpha \in (0,1)$. Then for any $2^k \leqslant n < 2^{k+1}$ $(k, n \in N)$ the inequality*

$$\|\sigma_n^{-\alpha}(f) - f\|_p \leqslant c(p, \alpha) \left\{ 2^{k\alpha} \omega(1/2^{k-1}, f)_p + \sum_{r=0}^{k-2} 2^{r-k} \omega(1/2^r, f)_p \right\}$$

holds true.

The following corollary is well known. For $p=\infty$, see [9] and for $p\in[1,\infty)$, see [4, 5].

Corollary 1. *Let $f(x)$ belong to $L^p[(0,1)]$ for some $p\in[1,\infty]$ and let*

$$2^{k\alpha}\omega(1/2^k, f)_p \to 0 \quad \text{as} \quad k\to\infty \quad (0<\alpha<1).$$

Then

$$\|\sigma_n^{-\alpha}(f)-f\|_p \to 0 \quad \text{as} \quad n\to\infty.$$

In case $p=\infty$ the sharpness of Corollary 1 has been proved by Tevzadze [9]. The following theorem shows that Corollary 1 cannot be improved in case $p=1$. In particular, we prove the following

Theorem 2. *For every $\alpha\in(0,1)$ there exists a function $f_0\in L^1([0,1])$ for which*

$$\omega(\delta, f_0)_1 = O(\delta^\alpha)$$

and

$$\overline{\lim_{n\to\infty}}\ \|\sigma_{2^n}^{-\alpha}(f_0)-f_0\|_1 < 0.$$

Corollary 2. *Let $\alpha\in(0,1)$. Then for every $q<1/(1-\alpha)$ there exists a function $f_0\in L^q([0,1])$ for which*

$$\overline{\lim_{n\to\infty}}\ \|\sigma_{2^n}^{-\alpha}(f_0)-f_0\|_1 > 0.$$

2. AUXILIARY RESULTS

Lemma 1 [2]. *Let $\alpha_1, \ldots, \alpha_n$ be real numbers. Then*

$$\frac{1}{n}\int_0^1 \left|\sum_{k=1}^n \alpha_k D_k(x)\right| dx \leqslant \frac{c}{\sqrt{n}}\left(\sum_{k=1}^n \alpha_k^2\right)^{1/2},$$

where c is an absolute constant.

Lemma 2 [4]. *Let $f\in L^p([0,1])$ for some $p\in[1,+\infty]$. Then for every $\alpha\in(0,1)$ the following estimations hold*

$$\frac{1}{A_n^{-\alpha}}\left\|\int_0^1 \sum_{\nu=2^{k-1}}^{2^k-1} A_{n-\nu}^{-\alpha} w_\nu(u)[f(\cdot \oplus u)-f(\cdot)]\,du\right\|_p \leqslant c(p,\alpha)\,\omega(1/2^{k-1}, f)_p\, 2^{k\alpha},$$

$$\frac{1}{A_n^{-\alpha}}\left\|\int_0^1 \sum_{\nu=2^k}^{n} A_{n-\nu}^{-\alpha} w_\nu(u)[f(\cdot \oplus u)-f(\cdot)]\,du\right\|_p \leqslant c(p,\alpha)\,\omega(1/2^{k}, f)_p\, 2^{k\alpha},$$

where $2^k \leqslant n < 2^{k+1}$.

LEMMA 3 [9]. *If* $\alpha \in (0,1)$ *and* $p \geqslant 2^m$ *then*

$$sign\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha} w_\nu(t)\right) = sign\, w_{2^m-1}(t), \qquad t \in (0,1).$$

LEMMA 4 [3, Chap. 10]. *Let* $1 \leqslant p < q < \infty$ *and* $f \in L^p([0,1])$. *If*

$$\sum_{n=1}^{\infty} n^{q/p-2}[\omega(1/n, f)_p]^q < \infty,$$

then $f \in L^q([0,1])$.

LEMMA 5. *Let* $f \in L^p([0,1])$ *for some* $p \in [1,+\infty]$. *Then for every* $\alpha \in (0,1)$ *the following estimation holds*

$$\frac{1}{A_n^{-\alpha}}\left\|\int_0^1 \sum_{\nu=0}^{2^{k-1}-1} A_{n-\nu}^{-\alpha} w_\nu(u)[f(\cdot \oplus u)-f(\cdot)]\,du\right\|_p \leqslant c(p,\alpha)\sum_{r=0}^{k-1} 2^{r-k}\omega(1/2^r, f)_p,$$

where $2^k \leqslant n < 2^{k+1}$.

Proof. Applying Abel's transformation, from (3) we get

$$\frac{1}{A_n^{-\alpha}}\left\|\int_0^1 \sum_{\nu=0}^{2^{k-1}-1} A_{n-\nu}^{-\alpha} w_\nu(u)[f(\cdot \oplus u)-f(\cdot)]\,du\right\|_p = \frac{1}{A_n^{-\alpha}}\left\|\int_0^1 \sum_{\nu=1}^{2^{k-1}} A_{n-\nu-1}^{-\alpha} w_{\nu-1}(u)[f(\cdot \oplus u)-f(\cdot)]\,du\right\|_p \tag{5}$$

$$\leqslant \frac{1}{A_n^{-\alpha}} \left\| \int_0^1 \sum_{\nu=1}^{2^{k-1}-1} A_{n-\nu-1}^{-\alpha-1} D_\nu(u)[f(\cdot \oplus u)-f(\cdot)]\,du \right\|_p$$
$$+\frac{1}{A_n^{-\alpha}} \left\| \int_0^1 A_{n-2^{k-1}-1}^{-\alpha} D_{2^{k-1}}(u)[f(\cdot \oplus u)-f(\cdot)]\,du \right\|_p$$
$$=I+II.$$

It is evident that

$$I=\frac{1}{A_n^{-\alpha}} \left\| \int_0^1 \sum_{r=0}^{k-2} \sum_{\nu=0}^{2^r-1} A_{n-\nu-2^r-1}^{-\alpha-1} D_{\nu+2^r}(u)[f(\cdot \oplus u)-f(\cdot)]\,du \right\|_p \tag{6}$$
$$\leqslant \frac{1}{A_n^{-\alpha}} \left\| \int_0^1 \sum_{r=1}^{k-2} \sum_{\nu=1}^{2^r-1} A_{n-\nu-2^r-1}^{-\alpha-1} D_{\nu+2^r}(u)[f(\cdot \oplus u)-f(\cdot)]\,du \right\|_p$$
$$+\frac{1}{A_n^{-\alpha}} \left\| \int_0^1 \sum_{r=0}^{k-2} A_{n-2^r-1}^{-\alpha-1} D_{2^r}(u)[f(\cdot \oplus u)-f(\cdot)]\,du \right\|_p$$
$$=I_1+I_2.$$

From the generalized Minkowski inequality, and by (1) and (4) we have

$$II \leqslant \frac{A_{n-2^{k-1}-1}^{-\alpha}}{A_n^{-\alpha}} 2^{k-1} \int_0^{1/2^{k-1}} \|f(\cdot \oplus u)-f(\cdot)\|_p\,du = O(\omega(1/2^{k-1}, f)_p), \tag{7}$$

$$I_2 \leqslant \frac{1}{A_n^{-\alpha}} \sum_{r=0}^{k-2} 2^r |A_{n-2^r-1}^{-\alpha-1}| \int_0^{1/2^r} \|f(\cdot \oplus u)-f(\cdot)\|_p\,du \tag{8}$$
$$=O\left(n^\alpha \sum_{r=0}^{k-2} \frac{\omega(1/2^r, f)_p}{(n-2^r-1)^{1+\alpha}} \right) = O\left(\frac{n^\alpha}{2^{k(1+\alpha)}} \sum_{r=0}^{k-2} \omega(1/2^r, f)_p \right)$$
$$=O\left(\sum_{r=0}^{k-2} 2^{r-k} \omega(1/2^r, f)_p \right).$$

Since

$$D_{\nu+2^r}(u)=D_{2^r}(u)+w_{2^r}(u)\,D_\nu(u), \tag{9}$$

for I_1, we write

$$I_1 \leqslant \frac{1}{A_n^{-\alpha}} \left\| \int_0^1 \sum_{r=1}^{k-2} \sum_{\nu=1}^{2^r-1} A_{n-\nu-2^r-1}^{-\alpha-1} D_\nu(u)\,w_{2^r}(u)[f(\cdot \oplus u)-f(\cdot)]\,du \right\|_p \tag{10}$$
$$+\frac{1}{A_n^{-\alpha}} \left\| \int_0^1 \sum_{r=1}^{k-2} \sum_{\nu=1}^{2^r-1} A_{n-\nu-2^r-1}^{-\alpha-1} D_{2^r}(u)[f(\cdot \oplus u)-f(\cdot)]\,du \right\|_p$$
$$=I_{11}+I_{12}.$$

The estimation of I_{12} is analogous to the estimation of I_2 and we have

(11) $$I_{12} = O\left(\sum_{r=1}^{k-2} 2^{r-k}\omega(1/2^r, f)_p\right).$$

For I_{11}, we write

(12) $$I_{11} \leqslant \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \left\| \sum_{l=0}^{2^r-1} \int_{l/2^r}^{(l+1)/2^r} \sum_{v=1}^{2^r-1} A_{n-v-2^r-1}^{-\alpha-1} D_v(u)\, w_{2^r}(u)] \right.$$
$$\left. f(\cdot \oplus u) - f(\cdot)]\, du \right\|_p$$
$$= \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \left\| \sum_{l=0}^{2^r-1} \sum_{v=1}^{2^r-1} A_{n-v-2^r-1}^{-\alpha-1} D_v\left(\frac{l}{2^r}\right) \int_{l/2^r}^{(l+1)/2^r} w_{2^r}(u) \right.$$
$$\left. [f(\cdot \oplus u) - f(\cdot)]\, du \right\|_p .$$

Since

$$w_{2^r}(u) = \begin{cases} 1, & \text{if } u \in \left[\frac{2l}{2^{r+1}}, \frac{2l+1}{2^{r+1}}\right) \\ -1, & \text{if } u \in \left[\frac{2l+1}{2^{r+1}}, \frac{2l+2}{2^{r+1}}\right) \end{cases}$$

and $t = u \oplus 1/2^{r+1}$ is a one-to-one mapping of $[2l/2^{r+1}, (2l+1)/2^{r+1})$ onto $[(2l+1)/2^{r+1}, (2l+2)/2^{r+1})$, we have

(13) $$\int_{l/2^r}^{(l+1)/2^r} w_{2^r}(u)[f(x \oplus u) - f(x)]\, du$$
$$= \int_{2l/2^{r+1}}^{(2l+1)/2^{r+1}} \left[f(x \oplus u) - f\left(x \oplus u \oplus \frac{1}{2^{r+1}}\right)\right] du.$$

After substituting (13) in (12) we obtain by (4) and Lemma 1 that

(14) $$I_{11} \leqslant \sum_{r=1}^{k-2} \left\| \frac{1}{A_n^{-\alpha}} \sum_{l=0}^{2^r-1} \sum_{v=1}^{2^r-1} A_{n-v-2^r-1}^{-\alpha-1} D_v\left(\frac{l}{2^r}\right) \right.$$
$$\left. \times \int_{2l/2^{r+1}}^{(2l+1)/2^r+1} \left[f(\cdot \oplus u) - f\left(\cdot \oplus u \oplus \frac{1}{2^{r+1}}\right)\right] du \right\|_p$$

$$\leqslant \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \sum_{l=0}^{2^r-1} \left| \sum_{v=1}^{2^r-1} A_{n-v-2^r-1}^{-\alpha-1} D_v\left(\frac{l}{2^r}\right) \right|$$

$$\times \int_{2l/2^{r+1}}^{(2l+1)/2^r+1} \left\| f(\cdot \oplus u) - f\left(\cdot \oplus u \oplus \frac{1}{2^{r+1}}\right) \right\|_p du$$

$$\leqslant \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \omega(1/2^r, f)_p \sum_{l=0}^{2^r-1} \left| \sum_{v=1}^{2^r-1} A_{n-v-2^r-1}^{-\alpha-1} D_v\left(\frac{l}{2^r}\right) \right| \int_{2l/2^{r+1}}^{(2l+1)/2^r+1} du$$

$$= \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \omega(1/2^r, f)_p \sum_{l=0}^{2^r-1} \int_{2l/2^{r+1}}^{(2l+1)/2^{r+1}} \left| \sum_{v=1}^{2^r-1} A_{n-v-2^r-1}^{-\alpha-1} D_v(u) \right| du$$

$$\leqslant \frac{1}{A_n^{-\alpha}} \sum_{r=1}^{k-2} \omega(1/2^r, f)_p \int_0^1 \left| \sum_{v=1}^{2^r-1} A_{n-v-2^r-1}^{-\alpha-1} D_v(u) \right| du$$

$$= O\left(n^\alpha \sum_{r=1}^{k-2} 2^r \omega(1/2^r, f)_p \frac{1}{2^{r/2}} \left(\sum_{v=1}^{2^r-1} (n-v-2^r-1)^{-2\alpha-2} \right)^{1/2} \right)$$

$$= O\left(\sum_{r=1}^{k-2} 2^{r-k} \omega(1/2^r, f)_p \right).$$

Combining (5)–(8), (10), (11), and (14) e receive the proof of Lemma 5. ∎

3. PROOFS OF MAIN RESULTS

Proof of Theorem 1. It is evident that

$$\sigma_n^{-\alpha}(f, x) - f(x) = \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{v=0}^{n} A_{n-v}^{-\alpha} w_v(x)[f(x \oplus u) - f(x)]\, du$$

$$= \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{v=0}^{2^{k-1}-1} A_{n-v}^{-\alpha} w_v(x)[f(x \oplus u) - f(x)]\, du$$

$$+ \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{v=2^{k-1}}^{2^k-1} A_{n-v}^{-\alpha} w_v(x)[f(x \oplus u) - f(x)]\, du$$

$$+ \frac{1}{A_n^{-\alpha}} \int_0^1 \sum_{v=2^k}^{n} A_{n-v}^{-\alpha} w_v(x)[f(x \oplus u) - f(x)]\, du$$

$$= I + II + III.$$

Since

$$\|\sigma_n^{-\alpha}(f, x) - f(x)\|_p \leqslant \|I\|_p + \|II\|_p + \|III\|_p.$$

From Lemma 2 and Lemma 5 the proof of Theorem 1 is complete. ∎

Proof of Theorem 2. We choose a monotonicaly increasing sequence of positive integers $\{n_k : k \geqslant 1\}$ such that

$$\frac{4}{2^{n_1\alpha}}\left(\frac{2^{\alpha}c_2(\alpha)}{c_1(\alpha)}+1\right) \leqslant \frac{1-\alpha}{2c_2(\alpha)} \tag{15}$$

and

$$n_{k-1}+1/\alpha < n_k, \qquad k \geqslant 2, \tag{16}$$

where

$$c_1(\alpha) = \inf_n \frac{A_n^{-\alpha}}{n^{-\alpha}}, \quad c_2(\alpha) = \sup_n \frac{A_n^{-\alpha}}{n^{-\alpha}}.$$

Let

$$f_0(x) = \sum_{k=1}^{\infty} f_k(x),$$

where

$$f_k(x) = \frac{1}{2^{n_k\alpha}} \sum_{j=2^{n_k-1}}^{2^{n_k}-1} w_j(x).$$

It is evident that $f_0 \in L_1([0, 1])$.
First we prove that

$$\omega(\delta, f_0)_1 = O(\delta^{\alpha}). \tag{17}$$

For every $\delta > 0$ there exists a positive integer k such that

$$1/2^{n_k} \leqslant \delta < 1/2^{n_{k-1}}.$$

Since $\omega(1/2^n, f)_p \leqslant 2\,\|f - S_{2^n}(f)\|_p$ $(n \in N)$ (see [3, Chap. 10]) we have

$$\omega(\delta, f_0)_1 \leqslant \omega\left(\frac{1}{2^{n_{k-1}}}, f_0\right)_1 \leqslant 2\,\|f_0 - S_{2^{n_{k-1}}}(f_0)\|_1 \tag{18}$$

$$\leqslant \sum_{j=k}^{\infty} \|f_j\|_1 \leqslant 2 \sum_{j=k}^{\infty} \frac{1}{2^{n_j\alpha}} \leqslant \frac{4}{2^{n_k\alpha}} \leqslant 4\delta^{\alpha}.$$

Next we shall prove that $\sigma_{2^{n_k}}^{-\alpha}(f_0)$ diverges in the metric of $L_1([0,1])$. It is clear that

$$\|\sigma_{2^{n_k}}^{-\alpha}(f_0)-f_0\|_1 \geqslant \|\sigma_{2^{n_k}}^{-\alpha}(f_0)\|_1-\|f_0\|_1 \tag{19}$$

$$\leqslant \|\sigma_{2^{n_k}}^{-\alpha}(f_k)\|_1-\sum_{l=1}^{k-1}\|\sigma_{2^{n_k}}^{-\alpha}(f_l)\|_1-\sum_{s=k+1}^{\infty}\|\sigma_{2^{n_k}}^{-\alpha}(f_s)\|_1-\sum_{j=1}^{\infty}\|f_j\|_1.$$

Since $\hat{f}_s(j)=0$, $j=0,1,\ldots,2^{n_k}$, $s=k+1,k+2,\ldots,$ we have

$$\sigma_{2^{n_k}}^{-\alpha}(f_s)=0. \tag{20}$$

Let $l\leqslant k-1$. Since

$$\hat{f}_l(j)=\begin{cases}2^{-n_l\alpha} & \text{for} \quad j=\overline{2^{n_l-1},2^{n_l}-1},\\ 0, & \text{for other} \quad j,\end{cases}$$

we obtain by Lemma 3 that

$$|\sigma_{2^{n_k}}^{-\alpha}(f_l,x)|=\frac{1}{A_{2^{n_k}}^{-\alpha}}\left|\sum_{j=0}^{2^{n_k}}A_{2^{n_k}-j}^{-\alpha}w_j(x)\,\hat{f}_l(j)\right|$$

$$=\frac{1}{A_{2^{n_k}}^{-\alpha}}\frac{1}{2^{n_l\alpha}}\left|\sum_{j=2^{n_l-1}}^{2^{n_l}-1}A_{2^{n_k}-j}^{-\alpha}w_j(x)\right|$$

$$\leqslant\frac{1}{A_{2^{n_k}}^{-\alpha}}\frac{1}{2^{n_l\alpha}}\left[\left|\sum_{j=0}^{2^{n_l}-1}A_{2^{n_k}-j}^{-\alpha}w_j(x)\right|+\left|\sum_{j=0}^{2^{n_l-1}-1}A_{2^{n_k}-j}^{-\alpha}w_j(x)\right|\right]$$

$$=\frac{1}{A_{2^{n_k}}^{-\alpha}}\frac{1}{2^{n_l\alpha}}\left[\sum_{j=0}^{2^{n_l}-1}A_{2^{n_k}-j}^{-\alpha}w_j(x)\,w_{2^{n_l}-1}(x)+\sum_{j=0}^{2^{n_l-1}-1}A_{2^{n_k}-j}^{-\alpha}w_j(x)\,w_{2^{n_l-1}-1}(x)\right].$$

Hence

$$\int_0^1|\sigma_{2^{n_k}}^{-\alpha}(f_l,x)|\,dx \tag{21}$$

$$\leqslant\frac{1}{A_{2^{n_k}}^{-\alpha}}\frac{1}{2^{n_l\alpha}}\sum_{j=0}^{2^{n_l}-1}A_{2^{n_k}-j}^{-\alpha}\int_0^1 w_j(x)\,w_{2^{n_l}-1}(x)\,dx$$

$$+\frac{1}{A_{2^{n_k}}^{-\alpha}}\frac{1}{2^{n_l\alpha}}\sum_{j=0}^{2^{n_l-1}-1}A_{2^{n_k}-j}^{-\alpha}\int_0^1 w_j(x)\,w_{2^{n_l-1}-1}(x)\,dx$$

$$= \frac{1}{A_{2^{n_k}}^{-\alpha}} \frac{1}{2^{n_l \alpha}} (A_{2^{n_k}-2^{n_l}+1}^{-\alpha} + A_{2^{n_k}-2^{n_l-1}+1}^{-\alpha}) \leqslant \frac{2c_2(\alpha)(2^{n_k}-2^{n_l})^{-\alpha}}{c_1(\alpha)\, 2^{-n_k\alpha}} \frac{1}{2^{n_l\alpha}}$$

$$\leqslant \frac{2c_2(\alpha)(2^{n_k}-2^{n_k-1})^{-\alpha}}{c_1(\alpha)\, 2^{-n_k\alpha}} \frac{1}{2^{n_l\alpha}} = \frac{2^{1+\alpha}c_2(\alpha)}{c_1(\alpha)} \frac{1}{2^{n_l\alpha}}.$$

From (16) and (21) we obtain

$$\sum_{l=1}^{k-1} \|\sigma_{2^{n_k}}^{-\alpha}(f_l)\|_1 \leqslant \frac{2^{1+\alpha}c_2(\alpha)}{c_1(\alpha)} \sum_{l=1}^{k-1} \frac{1}{2^{n_l\alpha}} \leqslant \frac{2^{2+\alpha}c_2(\alpha)}{c_1(\alpha)} \frac{1}{2^{n_1\alpha}}. \tag{22}$$

It is evident that

$$\sum_{j=1}^{\infty} \|f_j\|_1 \leqslant 2 \sum_{j=1}^{\infty} \frac{1}{2^{n_j\alpha}} \leqslant \frac{4}{2^{n_1\alpha}}. \tag{23}$$

Since

$$\hat{f}_k(j) = \begin{cases} 2^{-n_k\alpha}, & \text{for } \quad j = \overline{2^{n_k-1}, 2^{n_k}-1}, \\ 0, & \text{for other } \quad j, \end{cases}$$

and $\|g\|_1 \geqslant \hat{g}(j)$ $(j \in N, g \in L^1[0, 1])$ we have

$$\int_0^1 |\sigma_{2^{n_k}}^{-\alpha}(f_k, x)|\, dx \geqslant (\widehat{\sigma_{2^{n_k}}^{-\alpha}(f_k)})(2^{n_k}-1) = \frac{1}{A_{2^{n_k}}^{-\alpha}} \frac{1}{2^{n_k\alpha}} A_1^{-\alpha} \geqslant \frac{1-\alpha}{c_2(\alpha)}. \tag{24}$$

Owing to (15), (19), (20), (22), (23), and (24) we get

$$\|\sigma_{2^{n_k}}^{-\alpha}(f_0) - f_0\|_1 \geqslant \frac{1-\alpha}{c_2(\alpha)} - \frac{2^{2+\alpha}c_2(\alpha)}{c_1(\alpha)} \frac{1}{2^{n_1\alpha}} - \frac{4}{2^{n_1\alpha}}$$

$$\geqslant \frac{1-\alpha}{c_2(\alpha)} - \frac{1-\alpha}{2c_2(\alpha)} = \frac{1-\alpha}{2c_2(\alpha)} > 0.$$

Theorem 2 is proved. ∎

Proof of Corollary 2. Since

$$1 + q(\alpha - 1) > 0$$

and

$$\omega(\delta, f_0)_1 = O(\delta^\alpha),$$

we get

$$\sum_{n=1}^{\infty} n^{q-2}[\omega(1/n, f_0)_1]^q \leqslant \sum_{n=1}^{\infty} \frac{1}{n^{2+q(\alpha-1)}} < \infty.$$

Then $f_0 \in L^q([0, 1])$ follows from Lemma 4. ∎

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